Symbolic Kinematics for Linkages

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Abstract

This supplemental material provides details on the symbolic kinematics algorithm described in the main paper. In particular, we formulate three recursive reconstruction rules that allow us to treat all joints of a planar linkage in a unified manner. Furthermore, we provide the derivatives of these rules w.r.t. the initial joint locations.

1 Reconstruction Rules

We start by introducing rules for joints on fixed components and motors, then proceed to the reconstruction rule for general joints. We follow the same notation as in the main paper, denoting joint positions in the initial and stepped configurations with $\bar{\mathbf{x}}$ and $\mathbf{x}$, respectively. When taking derivatives, we follow the numerator-layout notation.

1.1 Joints on Fixed Components

As illustrated in Fig. 1 (left, middle), the positions $\mathbf{x}_k$ of joints and connectors on fixed components (in red) remain constant throughout the entire motion cycle of a given linkage. Hence, the *fixed joint rule* maps joints k onto themselves as

$$\mathbf{x}_k = \bar{\mathbf{x}}_k.$$

The state derivatives $\frac{\partial \mathbf{x}_k}{\partial \bar{\mathbf{x}}}$ are zero everywhere except for the block corresponding to $\bar{\mathbf{x}}_k$,

$$\frac{\partial \mathbf{x}_k}{\partial \bar{\mathbf{x}}_k} = \mathbf{I}_2 \ ,$$

where $\mathbf{I}_2$ denotes the 2×2 identity matrix.

1.2 Motors

Given the position $\mathbf{x}_j$ of a motor on the fixed component together with a motor angle ϕ_i as illustrated in Fig. 1 (right), we can reconstruct the positions of all joints k incident to j by rotating the vector $\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_j$ by ϕ about $\mathbf{x}_j$. The corresponding *motor rule* reads

$$\mathbf{x}_k = \mathbf{R}(\lambda\phi_i)\,(\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_j) + \mathbf{x}_j \ , \tag{1}$$

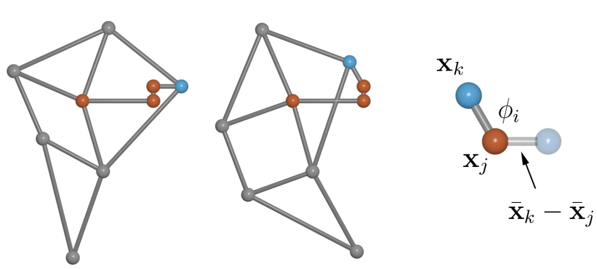

Figure 1: ***Fixed Joint and Motor Rules.*** *Initial and stepped configurations (left, middle) of a leg of the Jansen linkage: joints and connectors on a fixed component are shown in red, a joint incident to the motor in blue. Illustration of the motor rule (right) for a counterclockwise motion ($\lambda = 1$).*

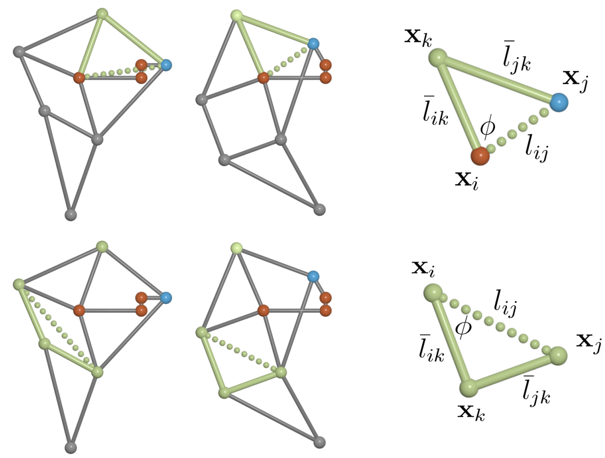

Figure 2: ***Triangle Rule.*** *We record triangle orientations in the initial configuration, preserving them to prevent singularities: we have a counterclockwise rotation ($\lambda = 1$, top) and a clockwise rotation ($\lambda = -1$, bottom).*

where the matrix

$$\mathbf{R}(\lambda\phi_i) = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \end{bmatrix} = \begin{bmatrix} c_i & -s_i \\ s_i & c_i \end{bmatrix} \quad \text{with} \quad \begin{matrix} c_i = \cos(\lambda\phi_i) \\ s_i = \sin(\lambda\phi_i) \end{matrix}$$

either encodes a clockwise or counterclockwise rotation, depending on the user-specified sign

$$\lambda = \begin{cases} 1 & \text{counterclockwise} \\ -1 & \text{clockwise.} \end{cases}$$

By default, we assume motors with uniform angular velocity, i.e., $\phi_i = i\Delta\phi$ with $\Delta\phi = \frac{2\pi}{n}$. However, the user can also specify the angles ϕ_i for n discrete samples $i = 0, \dots, n-1$ to achieve a non-uniform motor profile.

In order to obtain the state derivatives for joints incident to a motor, it is convenient to first spell out the individual components in Eq. 1,

$$\mathbf{x}_k = \begin{bmatrix} \mathbf{r}_1^T\,(\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_j) \\ \mathbf{r}_2^T\,(\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_j) \end{bmatrix} + \mathbf{x}_j,$$

then take derivatives w.r.t. positions in the initial configuration,

$$\frac{\partial \mathbf{x}_k}{\partial \bar{\mathbf{x}}} = \begin{bmatrix} \mathbf{r}_1^T \left(\frac{\partial \bar{\mathbf{x}}_k}{\partial \bar{\mathbf{x}}} - \frac{\partial \bar{\mathbf{x}}_j}{\partial \bar{\mathbf{x}}} \right) \\ \mathbf{r}_2^T \left(\frac{\partial \bar{\mathbf{x}}_k}{\partial \bar{\mathbf{x}}} - \frac{\partial \bar{\mathbf{x}}_j}{\partial \bar{\mathbf{x}}} \right) \end{bmatrix} + \frac{\partial \mathbf{x}_j}{\partial \bar{\mathbf{x}}}.$$

Note that $\frac{\partial \mathbf{x}_j}{\partial \bar{\mathbf{x}}}$ is readily available from state derivative evaluations of *fixed joint rules*. Similar to fixed joints, the only non-zero blocks in the state derivatives of $\bar{\mathbf{x}}_j$ and $\bar{\mathbf{x}}_k$ are $\frac{\partial \bar{\mathbf{x}}_j}{\partial \bar{\mathbf{x}}_j} = \mathbf{I}_2$ and $\frac{\partial \bar{\mathbf{x}}_k}{\partial \bar{\mathbf{x}}_k} = \mathbf{I}_2$, respectively.

1.3 Triangles

Once two joints i and j incident to an unknown joint k are known, we can reconstruct $\mathbf{x}_k$ by first scaling the vector $\mathbf{x}_j - \mathbf{x}_i$ to length $\bar{l}_{ik}$ (factor: $f = \frac{\bar{l}_{ik}}{l_{ij}}$), then rotating the resulting vector about $\mathbf{x}_i$ by an angle ϕ as illustrated in Fig. 2. The corresponding *triangle rule* follows as

$$\mathbf{x}_k = \mathbf{R}(\lambda\phi) f \left(\mathbf{x}_j - \mathbf{x}_i\right) + \mathbf{x}_i \ ,$$

where λ denotes a per-triangle sign that encodes the triangles' orientations. Note that, once determined by the ordering algorithm described in [Bächer et al. 2015], this sign stays constant throughout the motion cycle and is also preserved during all user edits (see main paper for details).

We use the law of cosines in order to obtain the rotation angle $\phi = \arccos(t)$ at joint i, where

$$t = \frac{t_n}{t_d} = \frac{l_{ij}^2 + \bar{l}_{ik}^2 - \bar{l}_{jk}^2}{2 l_{ij} \bar{l}_{ik}} \ .$$

The triangle's side lengths are $l_{ij} = ||\mathbf{x}_i - \mathbf{x}_j||$, $\bar{l}_{ik} = ||\bar{\mathbf{x}}_i - \bar{\mathbf{x}}_k||$, and $\bar{l}_{jk} = ||\bar{\mathbf{x}}_j - \bar{\mathbf{x}}_k||$. Note that the latter two stay constant throughout a motion cycle, whereas l_{ij} may vary.

The corresponding rotation matrix $\mathbf{R}(\lambda\phi)$ is

$$\begin{bmatrix} t & -\lambda\sqrt{1-t^2} \\ \lambda\sqrt{1-t^2} & t \end{bmatrix} ,$$

where we used $\cos(\arccos(t)) = t$, $\cos(\arcsin(t)) = \sqrt{1-t^2}$, and $\mathbf{R}(-\alpha) = \mathbf{R}^T(\alpha)$.

To obtain the state derivatives of $\mathbf{x}_k$, it is again convenient to introduce a matrix

$$\mathbf{Q}(\lambda\phi) = \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \end{bmatrix} = \begin{bmatrix} d & -o \\ o & d \end{bmatrix} = \mathbf{R}(\lambda\phi) f$$

with diagonal elements $d = tf$ and off-diagonal elements $o = \lambda\sqrt{1-t^2} f$. Separating components, we have

$$\mathbf{x}_k = \begin{bmatrix} \mathbf{q}_1^T \left(\mathbf{x}_j - \mathbf{x}_i\right) \\ \mathbf{q}_2^T \left(\mathbf{x}_j - \mathbf{x}_i\right) \end{bmatrix} + \mathbf{x}_i. \tag{2}$$

For the joint's state derivative $\frac{\partial \mathbf{x}_k}{\partial \bar{\mathbf{x}}}$, we then obtain

$$\begin{bmatrix} \frac{\partial}{\partial \bar{\mathbf{x}}} \left[\mathbf{q}_1^T \left(\mathbf{x}_j - \mathbf{x}_i\right)\right] \\ \frac{\partial}{\partial \bar{\mathbf{x}}} \left[\mathbf{q}_2^T \left(\mathbf{x}_j - \mathbf{x}_i\right)\right] \end{bmatrix} + \frac{\partial \mathbf{x}_i}{\partial \bar{\mathbf{x}}} \ ,$$

where we use the chain rule for the first

$$\frac{\partial}{\partial \bar{\mathbf{x}}} \left[\mathbf{q}_1^T \left(\mathbf{x}_j - \mathbf{x}_i\right)\right] = \left(\mathbf{x}_j - \mathbf{x}_i\right)^T \begin{bmatrix} \frac{\partial d}{\partial \bar{\mathbf{x}}} \\ -\frac{\partial o}{\partial \bar{\mathbf{x}}} \end{bmatrix} + \mathbf{q}_1^T \left(\frac{\partial \mathbf{x}_j}{\partial \bar{\mathbf{x}}} - \frac{\partial \mathbf{x}_i}{\partial \bar{\mathbf{x}}}\right)$$

and second row

$$\frac{\partial}{\partial \bar{\mathbf{x}}} \left[\mathbf{q}_2^T \left(\mathbf{x}_j - \mathbf{x}_i\right)\right] = \left(\mathbf{x}_j - \mathbf{x}_i\right)^T \begin{bmatrix} \frac{\partial o}{\partial \bar{\mathbf{x}}} \\ \frac{\partial d}{\partial \bar{\mathbf{x}}} \end{bmatrix} + \mathbf{q}_2^T \left(\frac{\partial \mathbf{x}_j}{\partial \bar{\mathbf{x}}} - \frac{\partial \mathbf{x}_i}{\partial \bar{\mathbf{x}}}\right)$$

of the (scaled) rotational part of Eq. 2.

The remaining expressions are readily derived using basic calculus:

$$\frac{\partial d}{\partial \bar{\mathbf{x}}} = \frac{\partial t}{\partial \bar{\mathbf{x}}} f + t \frac{\partial f}{\partial \bar{\mathbf{x}}}$$

$$\frac{\partial o}{\partial \bar{\mathbf{x}}} = -\lambda \frac{t}{\sqrt{1-t^2}} \frac{\partial t}{\partial \bar{\mathbf{x}}} f + \lambda\sqrt{1-t^2} \frac{\partial f}{\partial \bar{\mathbf{x}}}$$

$$\frac{\partial f}{\partial \bar{\mathbf{x}}} = \frac{1}{l_{ij}^2} \left[\frac{\partial \bar{l}_{ik}}{\partial \bar{\mathbf{x}}} l_{ij} - \bar{l}_{ik} \frac{\partial l_{ij}}{\partial \bar{\mathbf{x}}}\right]$$

$$\frac{\partial t}{\partial \bar{\mathbf{x}}} = \frac{1}{t_d} \left[\frac{\partial t_n}{\partial \bar{\mathbf{x}}} t_d - t_n \frac{\partial t_d}{\partial \bar{\mathbf{x}}}\right]$$

$$\frac{\partial l_{ij}}{\partial \bar{\mathbf{x}}} = \left(\frac{\mathbf{x}_i - \mathbf{x}_j}{||\mathbf{x}_i - \mathbf{x}_j||}\right)^T \left[\frac{\partial \mathbf{x}_i}{\partial \bar{\mathbf{x}}} - \frac{\partial \mathbf{x}_j}{\partial \bar{\mathbf{x}}}\right]$$

For the vectors $\frac{\partial \bar{l}_{ik}}{\partial \bar{\mathbf{x}}}$ and $\frac{\partial \bar{l}_{jk}}{\partial \bar{\mathbf{x}}}$, the only non-zero blocks are

$$\frac{\partial \bar{l}_{ik}}{\partial \bar{\mathbf{x}}_i} = -\frac{\partial \bar{l}_{ik}}{\partial \bar{\mathbf{x}}_k} = \left(\frac{\mathbf{x}_i - \mathbf{x}_k}{||\mathbf{x}_i - \mathbf{x}_k||}\right)^T$$

and

$$\frac{\partial \bar{l}_{jk}}{\partial \bar{\mathbf{x}}_j} = -\frac{\partial \bar{l}_{jk}}{\partial \bar{\mathbf{x}}_k} = \left(\frac{\mathbf{x}_j - \mathbf{x}_k}{||\mathbf{x}_j - \mathbf{x}_k||}\right)^T ,$$

respectively. Finally, state derivatives of $\mathbf{x}_i$ and $\mathbf{x}_j$ are obtained through recursive evaluations of previous rules.

References

Bächer, M., Coros, S., and Thomaszewski, B. 2015. LinkEdit: Interactive linkage editing using symbolic kinematics. *ACM Trans. Graph. (Proc. SIGGRAPH) 34*, 4.